Lecture 10

Momentum, Complex Poynting's Theorem, Lossless Condition, Energy Density

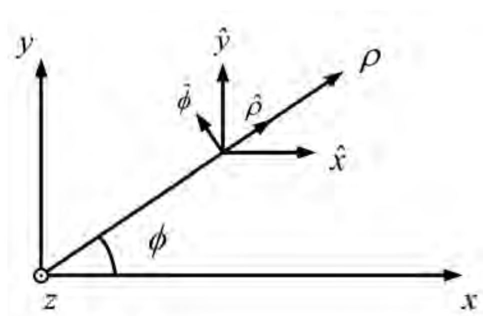

Figure 10.1: The local coordinates used to describe a circularly polarized wave: In cartesian and polar coordinates.

In the last lecture, we study circularly polarized waves as well as linearly polarized waves. In addition, these waves can carry power giving rise to power flow. But in addition to carrying power, a travelling wave also has a momentum: for a linearly polarized wave, it carries linear momention in the direction of the propagation of the traveling wave. But for a circularly polarized wave, it carries angular momentum as well.

We have studied complex power and the complex Poynting's theorem in the frequency domain with phasors in the previous lectures. Here, we will derive the lossless conditions for the permittivity and permeability tensors. As we have shown in the instantaneous Poynting's theorem, energy density is well defined for a lossless dispersionless medium, but we will learn that it assumes a different formula when the medium is dispersive.

10.1 Spin Angular Momentum and Cylindrical Vector Beam

In this section, we will study the spin angular momentum of a circularly polarized (CP) wave. It is to be noted that in cylindrical coordinates, as shown in Figure 10.1, $\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$, $\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$, then a CP field is proportional to

$$(\hat{x} \pm j\hat{y}) = \hat{\rho}e^{\pm j\phi} \pm j\hat{\phi}e^{\pm j\phi} = e^{\pm j\phi}(\hat{\rho} \pm \hat{\phi}) \tag{10.1.1}$$

Therefore, with the $e^{\pm j\phi}$ dependence, the $\hat{\rho}$ and $\hat{\phi}$ of a CP is also an azimuthal traveling wave in the $\hat{\phi}$ direction in addition to being a traveling wave $e^{-j\beta z}$ in the $\hat{z}$ direction. This is obviated by rewriting

$$e^{-j\phi} = e^{-jk_\phi \rho\phi} \tag{10.1.2}$$

where $k_\phi = 1/\rho$ is the azimuthal wave number, and $\rho\phi$ is the arc length traversed by the azimuthal wave. Notice that the wavenumber k_ϕ is dependent on ρ: the larger the ρ, the smaller is k_ϕ, and hence, the larger the azimuthal wavelength. Thus, the wave possesses angular momentum called the spin angular momentum (SAM), just as a traveling wave $e^{-j\beta z}$ possesses linear angular momentum in the $\hat{z}$ direction.

In optics research, the generation of cylindrical vector beam is in vogue. Figure 10.2 shows a method to generate such a beam. A CP light passes through a radial analyzer that will only allow the radial component of (10.1.1) to be transmitted. Then a spiral phase element (SPE) compensates for the $\exp(\pm j\phi)$ phase shift in the azimuthal direction. Finally, the light is a cylindrical vector beam which is radially polarized without spin angular momentum. Such a beam has been found to have nice focussing property, and hence, has aroused researchers' interest in the optics community [83].

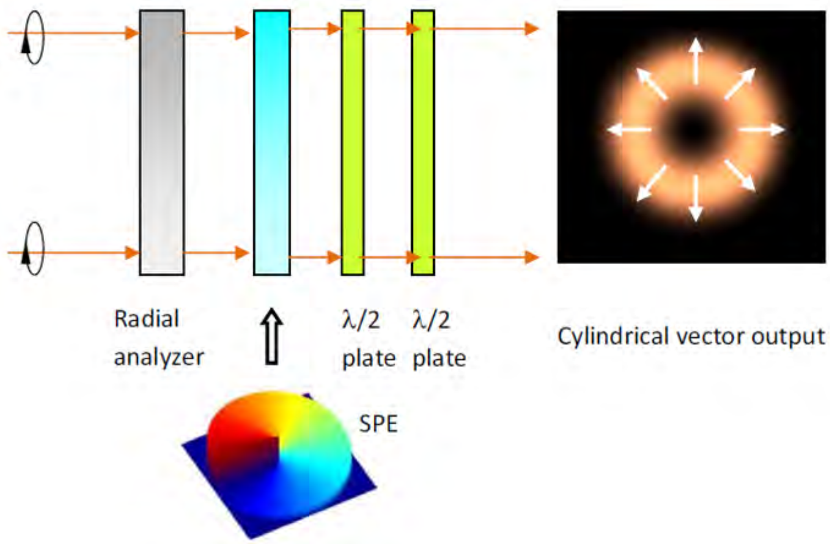

Figure 10.2: A cylindrical vector beam can be generated experimentally. The spiral phase element (SPE) compensates for the $\exp(\pm j\phi)$ phase shift (courtesy of Zhan, Q. [83]). The half-wave plate rotates the polarization of a wave by 90 degrees.

10.2 Momentum Density of Electromagnetic Field

We have seen that a traveling wave carries power and has energy density associated with it. In other words, the moving or traveling energy density gives rise to power flow. It turns out that a traveling wave also carries a momentum with it. The momentum density of electromagnetic field is given by

$$\mathbf{G} = \mathbf{D} \times \mathbf{B} \tag{10.2.1}$$

also called the momentun density vector. With it, one can derive momentum conservation theorem [33, p. 59] [47]. The derivation is rather long, but we will justify the above formula and simplify the derivation using the particle or corpuscular nature of light or electromagnetic field. The following derivation is only valid for plane waves.

It has been long known that electromagnetic energy is carried by photon, associated with a packet of energy given by $\hbar\omega$. It is also well known that a photon has momentum given by $\hbar k$. Assuming that there are photons, with density of N photons per unit volume, streaming through space at the velocity of light c. Then the power flow associated with these streaming photons is given by

$$\mathbf{E} \times \mathbf{H} = \hbar\omega N c \hat{z} \tag{10.2.2}$$

Assuming that the plane wave is propagating in the z direction. Using $k = \omega/c$, we can rewrite the above more suggestively as

$$\mathbf{E} \times \mathbf{H} = \hbar\omega N c\hat{z} = \hbar k N c^2 \hat{z} \tag{10.2.3}$$

where $k = \omega/c$. Defining the momentum density vector to be

$$\mathbf{G} = \hbar k N \hat{z} \tag{10.2.4}$$

From the above, we deduce that

$$\mathbf{E} \times \mathbf{H} = \mathbf{G}c^2 = \frac{1}{\mu\varepsilon}\mathbf{G} \tag{10.2.5}$$

Or the above can be rewritten as

$$\mathbf{G} = \mathbf{D} \times \mathbf{B} \tag{10.2.6}$$

where $\mathbf{D} = \varepsilon\mathbf{E}$, and $\mathbf{B} = \mu\mathbf{H}$.[1]

10.3 Complex Poynting's Theorem and Lossless Conditions

10.3.1 Complex Poynting's Theorem

It has been previously shown that the vector $\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$ has a dimension of watts/m^2 which is that of power density. Therefore, it is associated with the direction of power flow [33, 47]. As has been shown for time-harmonic field, a time average of this vector can be defined as

$$\langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)\, dt. \tag{10.3.1}$$

Given the phasors of time harmonic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$, namely, $\mathbf{E}(\mathbf{r}, \omega)$ and $\mathbf{H}(\mathbf{r}, \omega)$ respectively, we can show that

$$\langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \frac{1}{2} \Re e\{\mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}^*(\mathbf{r}, \omega)\}. \tag{10.3.2}$$

Here, the vector $\mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}^*(\mathbf{r}, \omega)$, as previously discussed, is also known as the complex Poynting's vector. If we define the instantaneous Poynting's vector to be

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \tag{10.3.3}$$

and the complex Poynting's vector to be

$$\underset{\sim}{\mathbf{S}}(\mathbf{r}, \omega) = \mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}^*(\mathbf{r}, \omega) \tag{10.3.4}$$

[1]The author is indebted to Wei SHA for this simple derivation.

then for time-harmonic fields,

$$\langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\Re e\left\{\underset{\sim}{\mathbf{S}}(\mathbf{r},\omega)\right\} \tag{10.3.5}$$

The above is often the source of confusion in the definition of Poynting's vector.

In the above definition of complex Poynting's vector and its aforementioned property, and that is has dimension of power density, we will study its conservative property. To do so, we take its divergence and use the appropriate vector identity to obtain[2]

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = \mathbf{H}^*\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}^*. \tag{10.3.6}$$

Next, using Maxwell's equations for $\nabla\times\mathbf{E}$ and $\nabla\times\mathbf{H}^*$, namely

$$\nabla\times\mathbf{E} = -j\omega\mathbf{B} \tag{10.3.7}$$

$$\nabla\times\mathbf{H}^* = -j\omega\mathbf{D}^* + \mathbf{J}^* \tag{10.3.8}$$

and the constitutive relations for anisotropic media that

$$\mathbf{B} = \overline{\boldsymbol{\mu}}\cdot\mathbf{H},\quad \mathbf{D}^* = \overline{\varepsilon}^*\cdot\mathbf{E}^* \tag{10.3.9}$$

we have

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = -j\omega\,\mathbf{H}^*\cdot\mathbf{B} + j\omega\,\mathbf{E}\cdot\mathbf{D}^* - \mathbf{E}\cdot\mathbf{J}^* \tag{10.3.10}$$

$$= -j\omega\,\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\mathbf{H} + j\omega\,\mathbf{E}\cdot\overline{\varepsilon}^*\cdot\mathbf{E}^* - \mathbf{E}\cdot\mathbf{J}^*. \tag{10.3.11}$$

The above is also known as the complex Poynting's theorem. It can also be written in an integral form using Gauss' divergence theorem, namely,

$$\int_S d\mathbf{S}\cdot(\mathbf{E}\times\mathbf{H}^*) = -j\omega\int_V dV(\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\mathbf{H} - \mathbf{E}\cdot\overline{\varepsilon}^*\cdot\mathbf{E}^*) - \int_V dV\mathbf{E}\cdot\mathbf{J}^*. \tag{10.3.12}$$

where S is the surface bounding the volume V.

10.3.2 Lossless Conditions

For a region V that is lossless and source-free, $\mathbf{J} = 0$. There should be no net time-averaged power-flow out of or into this region V. Therefore,

$$\Re e\int_S d\mathbf{S}\cdot(\mathbf{E}\times\mathbf{H}^*) = 0, \tag{10.3.13}$$

Because of the above, and energy conservation, the real part of the right-hand side of (10.3.11), without the $\mathbf{E}\cdot\mathbf{J}^*$ term, must also be zero. In other words, the right-hand side of (10.3.11) should be purely imaginary. Thus

$$\int_V dV(\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\mathbf{H} - \mathbf{E}\cdot\overline{\varepsilon}^*\cdot\mathbf{E}^*) \tag{10.3.14}$$

[2]The product rule for derivative will be used, and we will drop the argument $\mathbf{r},\omega$ for the phasors in our discussion next as they will be implied.

must be a real quantity.

Other than the possibility that the above is zero, the general requirement for (10.3.14) to be real for arbitrary $\mathbf{E}$ and $\mathbf{H}$, is that $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ and $\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*$ are real quantities. This is only possible if $\overline{\boldsymbol{\mu}}$ is hermitian.[3] Therefore, the conditions for anisotropic media to be lossless are

$$\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^\dagger, \qquad \overline{\boldsymbol{\varepsilon}} = \overline{\boldsymbol{\varepsilon}}^\dagger, \tag{10.3.15}$$

requiring the permittivity and permeability tensors to be hermitian. If this is the case, (10.3.14) is always real for arbitrary $\mathbf{E}$ and $\mathbf{H}$, and (10.3.13) is true, implying a lossless region V. Notice that for an isotropic medium, $\overline{\boldsymbol{\mu}} \to \mu$ and $\overline{\boldsymbol{\varepsilon}} \to \varepsilon$, this lossless conditions reduce simply to that $\Im m(\mu) = 0$ and $\Im m(\varepsilon) = 0$, or that μ and ε are pure real quantities. Looking back, many of the effective permittivities or dielectric constants that we have derived using the Drude-Lorentz-Sommerfeld model cannot be lossless when the friction term is nonzero. Looking at the formula for χ as given by (8.3.17), it cannot be real, and hence, it corresponds to a lossy medium.

For a lossy medium which is conductive, we may define $\mathbf{J} = \overline{\boldsymbol{\sigma}} \cdot \mathbf{E}$ where $\overline{\boldsymbol{\sigma}}$ is a general conductivity tensor. In this case, equation (10.3.12), after combining the last two terms, may be written as

$$\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega \int_V dV \left[\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \left(\overline{\boldsymbol{\varepsilon}}^* + \frac{j\overline{\boldsymbol{\sigma}}^*}{\omega} \right) \cdot \mathbf{E}^* \right] \tag{10.3.16}$$

$$= -j\omega \int dV [\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \tilde{\overline{\boldsymbol{\varepsilon}}}^* \cdot \mathbf{E}^*], \tag{10.3.17}$$

where $\tilde{\overline{\boldsymbol{\varepsilon}}} = \overline{\boldsymbol{\varepsilon}} - \frac{j\overline{\boldsymbol{\sigma}}}{\omega}$ which is the general complex permittivity tensor. In this manner, (10.3.17) has the same structure as the source-free Poynting's theorem. Notice here that the complex permittivity tensor $\tilde{\overline{\boldsymbol{\varepsilon}}}$ is clearly non-hermitian corresponding to a lossy medium.

For a lossless medium without the source term, by taking the imaginary part of (10.3.12), we arrive at

$$\Im m \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = -\omega \int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*), \tag{10.3.18}$$

The left-hand side of the above is the reactive power coming out of the volume V, and hence, the right-hand side can be interpreted as reactive power as well. It is to be noted that $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ and $\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*$ are not to be interpreted as stored energy density when the medium is dispersive. The correct expressions for stored energy density in dispersive media will be derived in the next section.

But, the quantity $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ for lossless, dispersionless media is associated with the time-averaged energy density stored in the magnetic field, while the quantity $\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*$ for lossless

[3] $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ is real only if its complex conjugate, or conjugate transpose is itself. Using some details from matrix algebra that $(\overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}})^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$, implies that (in physics notation, the transpose of a vector is implied in a dot product) $(\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H})^\dagger = (\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \mathbf{H}^*)^t = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \mathbf{H} = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$. The last equality in the above is possible only if $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^\dagger$ or that $\overline{\boldsymbol{\mu}}$ is hermitian.

dispersionless media is associated with the time-averaged energy density stored in the electric field. Then, for lossless, dispersionless, source-free media, then the right-hand side of the above can be interpreted as stored energy density. Hence, the reactive power is proportional to the time rate of change of the difference of the time-averaged energy stored in the magnetic field and the electric field.

10.4 Energy Density in Dispersive Media[4]

A dispersive medium alters our concept of what the formula energy density is.[5] In order to derive the new formula, we assume that the field has complex ω dependence in $e^{j\omega t}$, where $\omega = \omega' - j\omega''$, rather than real ω dependence. In other words, the field is not time-harmonic anymore.

We take the divergence of the complex power for fields with such time dependence, and let $e^{j\omega t}$ be attached to the field. So $\mathbf{E}(t)$ and $\mathbf{H}(t)$ are complex field but not exactly like phasors since they are not truly time harmonic. In other words, we let

$$\mathbf{E}(\mathbf{r}, t) = \underset{\sim}{\mathbf{E}}(\mathbf{r}, \omega)e^{j\omega t}, \quad \mathbf{H}(\mathbf{r}, t) = \underset{\sim}{\mathbf{H}}(\mathbf{r}, \omega)e^{j\omega t} \tag{10.4.1}$$

The above, just like phasors, can be made to satisfy Maxwell's equations where the time derivative becomes $j\omega$ but with complex ω. We can study the quantity $\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}^*(\mathbf{r}, t)$ which has the unit of power density. In the real ω case, their time dependence will exactly cancel each other and this quantity becomes complex power again. But here, the field is quasi-time-harmonic, and their time dependences do not cancel because of the complex ω. Hence,

$$\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = \mathbf{H}^*(t) \cdot \nabla \times \mathbf{E}(t) - \mathbf{E}(t) \cdot \nabla \times \mathbf{H}^*(t) \tag{10.4.2}$$

Maxwell's equations for this quasi-time-harmonic fields, when ω is complex, become

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \tag{10.4.3}$$

$$\nabla \times \mathbf{H}^* = -j\omega^*\mathbf{D}^* + \mathbf{J}^* \tag{10.4.4}$$

Using them in the above, we arrive at

$$\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = -\mathbf{H}^*(t) \cdot j\omega\mu\mathbf{H}(t) + \mathbf{E}(t) \cdot j\omega^*\varepsilon^*\mathbf{E}^*(t) \tag{10.4.5}$$

where Maxwell's equations have been used to substitute for $\nabla \times \mathbf{E}(t)$ and $\nabla \times \mathbf{H}^*(t)$. The space dependence of the field is implied, and we assume a source-free medium so that $\mathbf{J} = 0$.

If $\mathbf{E}(t) \sim e^{j\omega t}$, then, due to ω being complex, now $\mathbf{H}^*(t) \sim e^{-j\omega^* t}$, and the term like $\mathbf{E}(t) \times \mathbf{H}^*(t)$ is not truly time independent but becomes

$$\mathbf{E}(t) \times \mathbf{H}^*(t) \sim e^{j(\omega-\omega^*)t} = e^{2\omega'' t} \tag{10.4.6}$$

[4]The derivation in this section is complex, but worth the pain, since this knowledge was not discovered until the 1960s.

[5]The derivation here is inspired by H.A. Haus, Electromagnetic Noise and Quantum Optical Measurements [84]. Generalization to anisotropic media is given by W.C. Chew, Lectures on Theory of Microwave and Optical Waveguides [85].

And each of the terms above will have similar time dependence. Writing (10.4.2) more explicitly, by letting $\omega = \omega' - j\omega''$, we have

$$\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = -j(\omega' - j\omega'')\mu(\omega)|\mathbf{H}(t)|^2 + j(\omega' + j\omega'')\varepsilon^*(\omega)|\mathbf{E}(t)|^2 \tag{10.4.7}$$

Assuming that $\omega'' \ll \omega'$, or that the field is quasi-time-harmonic, we can let, after using Taylor series approximation, that

$$\mu(\omega' - j\omega'') \cong \mu(\omega') - j\omega'' \frac{\partial \mu(\omega')}{\partial \omega'}, \quad \varepsilon(\omega' - j\omega'') \cong \varepsilon(\omega') - j\omega'' \frac{\partial \varepsilon(\omega')}{\partial \omega'} \tag{10.4.8}$$

Using (10.4.8) in (10.4.7), and collecting terms of the same order, and ignoring $(\omega'')^2$ terms, gives[6]

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] \cong & - j\omega'\mu(\omega')|\mathbf{H}(t)|^2 + j\omega'\varepsilon^*(\omega')|\mathbf{E}(t)|^2 \\ & - \omega''\mu(\omega')|\mathbf{H}(t)|^2 - \omega'\omega''\frac{\partial \mu}{\partial \omega'}|\mathbf{H}(t)|^2 \\ & - \omega''\varepsilon^*(\omega')|\mathbf{E}(t)|^2 - \omega'\omega''\frac{\partial \varepsilon^*}{\partial \omega'}|\mathbf{E}(t)|^2\end{aligned} \tag{10.4.9}$$

The above can be rewritten as

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] \cong & -j\omega' \left[\mu(\omega')|\mathbf{H}(t)|^2 - \varepsilon^*(\omega')|\mathbf{E}(t)|^2\right] \\ & - \omega'' \left[\frac{\partial \omega'\mu(\omega')}{\partial \omega'}|\mathbf{H}(t)|^2 + \frac{\partial \omega'\varepsilon^*(\omega')}{\partial \omega'}|\mathbf{E}(t)|^2\right]\end{aligned} \tag{10.4.10}$$

The above approximation is extremely good when $\omega'' \ll \omega'$. For a lossless medium, $\varepsilon(\omega')$ and $\mu(\omega')$ are purely real, and the first term of the right-hand side is purely imaginary while the second term is purely real. In the limit when $\omega'' \to 0$, when half the imaginary part of the above equation is taken, we have

$$\nabla \cdot \frac{1}{2}\Im m\,[\mathbf{E} \times \mathbf{H}^*] = -\omega' \left[\frac{1}{2}\mu|\mathbf{H}|^2 - \frac{1}{2}\varepsilon|\mathbf{E}|^2\right] \tag{10.4.11}$$

The left-hand side and right-hand side of the above now can be interpreted as reactive power, something we have learnt in complex Poynting's theorem previously discussed.

When half the real part of (10.4.10) is taken, we obtain some new terms,

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*] = -\frac{\omega''}{2} \left[\frac{\partial \omega'\mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega'\varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] \tag{10.4.12}$$

The left-hand side of the above has the physical meaning of time-average power density when $\omega" \to 0$. Since the right-hand side has time dependence of $e^{2\omega'' t}$, when $\omega" \neq 0$, it can be written as

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*] = -\frac{\partial}{\partial t}\frac{1}{4} \left[\frac{\partial \omega'\mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega'\varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] = -\frac{\partial}{\partial t}\langle W_T \rangle \tag{10.4.13}$$

[6] This is the general technique of perturbation expansion [46].

The above is a restatement of that for a weakly time-harmonic system, the divergence of the time-average power density on the left-hand side is proportional to the time variation of the store energy on the right-hand side. Therefore, the time-average stored energy density can be identified as

$$\langle W_T \rangle = \frac{1}{4} \left[\frac{\partial \omega' \mu}{\partial \omega'} |\mathbf{H}|^2 + \frac{\partial \omega' \varepsilon}{\partial \omega'} |\mathbf{E}|^2 \right] \tag{10.4.14}$$

For a non-dispersive medium, μ and ε are independent of frequency, the above reverts back to a familiar expression,

$$\langle W_T \rangle = \frac{1}{4} \left[\mu |\mathbf{H}|^2 + \varepsilon |\mathbf{E}|^2 \right] \tag{10.4.15}$$

which is what we have derived before.

In the above analysis, we have used a quasi-time-harmonic signal with $\exp(j\omega t)$ dependence. In the limit when $\omega'' \to 0$, this signal reverts back to a time-harmonic signal, and to our usual interpretation of complex power. However, by assuming the frequency ω to have a small imaginary part ω'', it forces the stored energy to grow very slightly, and hence, power has to be supplied to maintain the growth of this stored energy. By so doing, and use of energy conservation, it allows us to identify the expression for energy density for a dispersive medium. These expressions for energy density were not discovered until 1960 by Brillouin [86], as energy density times group velocity should be power flow. More discussion on this topic can be found in Jackson [47].

It is to be noted that if the same analysis is used to study the energy storage in a capacitor or an inductor, the energy storage formulas have to be modified accordingly if the capacitor or inductor is frequency dependent.